Conditional Probability & Bayes' Theorem

Conditional Probability

The conditional probability of E given F is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{|E \cap F|}{|F|}$$

*As long as the distribution is uniform

Randomly generate a bit string of length four. Each outcome is equally likely.

What is the probability that it contains at least two consecutive 0's?

$S = \{0000, 0001, 0010, \ldots, 1111\}$ $|S| = 16$

$E = \{0000, 0001, 0010, 0011, 0100, 1000, 1001, 1100\}$ $|E| = 8$

$P(E) = \frac{8}{16} = \frac{1}{2}$

What if we know the first bit is a 0? Does that change the probability?

Now we are looking for the probability of two consecutive 0's *given* the first bit is a 0.

We have a different sample space. Now our possible outcomes only include those that start with a 0.

F = first bit is a 0, $|F| = 8$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$P(E|F) = \frac{\frac{5}{16}}{\frac{1}{2}} = \frac{5}{8}$$

Conditional Probability

The conditional probability of E given F is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{|E \cap F|}{|F|}$$

If we flip a fair coin 3 times, there are 8 possible outcomes:

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

The probability of each outcome is $\frac{1}{8}$. The size of the sample space $|S|$ is 8.

What is the probability of getting an odd number of tails?

$E = \{HHT, HTH, THH, TTT\}$, so $P(E) = \frac{|E|}{|S|} = \frac{4}{8} = \frac{1}{2}$

If we know the first coin flip is a tail, does that change the probability of getting an odd number of tails?

Our sample space has changed. Let's call the new sample space F. Then, $F = \{THH, THT, TTH, TTT\}$

Our new event space, given F, is everything that is in E and F, which is $E \cap F = \{THH, TTT\}$.

$$P(E \mid F) = \frac{|E \cap F|}{|F|} = \frac{2}{4} = \frac{1}{2}$$

The probability did not change. Why not?

Independent Events

If $P(E \mid F) = P(E)$, then E and F are independent events.

In other words, the probability of event E does not change based on event F.

The occurrence of event F does not give any information about event E.

We can also say:

Two events are independent if

$P(E \mid F) = P(E)$

$P(F \mid E) = P(F)$

$P(E \cap F) = P(E) \cdot P(F)$

These three conditions are all equivalent.

As a class

Additional Exercises

- 12.3.1 – Whiteboard
- 12.3.2 – Using Python

Additional Exercises

12.3.3

12.3.4

12.3.6

Mutual independence

$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdots P(E_n)$

Example:

n throws of a die where each event E_i depends only on the outcome of the i^{th} throw.

Flip a coin three times. E_i is the event that the i^{th} throw is a Head.

$$S = \{(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)\}$$

$$E_1 = \{(\boldsymbol{H}HH), (\boldsymbol{H}HT), (\boldsymbol{H}TH), (\boldsymbol{H}TT)\}$$
$$E_2 = \{(H\boldsymbol{H}H), (H\boldsymbol{H}T), (T\boldsymbol{H}H), (T\boldsymbol{H}T)\}$$
$$E_3 = \{(HH\boldsymbol{H}), (HT\boldsymbol{H}), (TH\boldsymbol{H}), (TT\boldsymbol{H})\}$$

$$E_1 \cap E_2 \cap E_3 = \{(HHH)\}$$

These events are mutually independent because a successful outcome in one event does not depend on the other events.

$P(E_1 \cap E_2 \cap E_3) = \frac{1}{8}$ $\quad$ $P(E_1) = P(E_2) = P(E_3) = \frac{1}{2}$ $\quad$ $P(E_1) \cdot P(E_2) \cdot P(E_3) = \frac{1}{8}$

$$\boldsymbol{P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2) \cdot P(E_3)}$$