Generating Permutations and Combinations

Review

What is the difference between a permutation and a subset/combination?

Generating Permutations

Last week, we learned to count the number of permutations of a set.

Sometimes, we want to list or ***generate*** all the permutations of a set, not just count them.

What are some reasons we might want to do this?

- Finding a Hamiltonian Cycle in a graph
- Planning a road trip to visit every state capital. What are the possible routes?
- Traveling Salesman problem. What is the optimal way to visit every city on the route and get back home?
- Delivering packages. What is the best order to deliver them in to minimize the total distance traveled by the driver?

Lexicographical Order

How many permutations are there for the set {a,b,c,d}?

$4! = 24$

How many of them start with the letter "a"?

$3! = 6$

List them all in lexicographical order:

abcd
abdc
acbd
acdb
adcb
adbc

Lexicographic order is a way of ordering n-tuples in which two n-tuples are compared according to the first entry where they differ. This is called "alphabetical order" when using letters/words.

Generating permutations

Finding the next permutation in lexicographical order:

1. Search the list from right to left and find the first item that is smaller than the previous item. Call this item x.
2. Swap x with the smallest item that is greater than x in the part of the list that comes after x.
3. Reorder the rest of the list in increasing order by reversing this part of the list.

What comes after this permutation?

abdec

abdec   Step 1: x = d

abedc   Step 2: Swap d and e

abecd   Step 3: Reverse the rest of the list

Lexicographical Order

List the first 10 permutations of the set {1,2,3,4,5} in lexicographical order, then compare with a partner.

12345
12354
12435
12453
12534
12543
13245
13254
13425
13452

1. Search the list from right to left and find the first item that is smaller than the previous item. Call this item x.
2. Swap x with the smallest number that is greater than x in the part of the list that comes after x.
3. Reorder the rest of the list in increasing order by reversing this part of the list.

How many total permutations are there?

$5! = 120$

Binomial Theorem

Whiteboard discussion

Pascal's Triangle

Take a moment to create Pascal's Triangle up to the 7th row.

Now use Pascal's Triangle to answer these questions:

How many ways can you choose 2 elements from a set of 7 things? 21

How many ways can you choose 3 elements from a set of 7 things? 35

How many ways can you choose 4 elements from a set of 5 things? 5

Pigeonhole Principle

If a function has a domain of size $n + 1$ or more, and a codomain of size n or less, then there are two elements in the domain that map to the same element in the codomain.

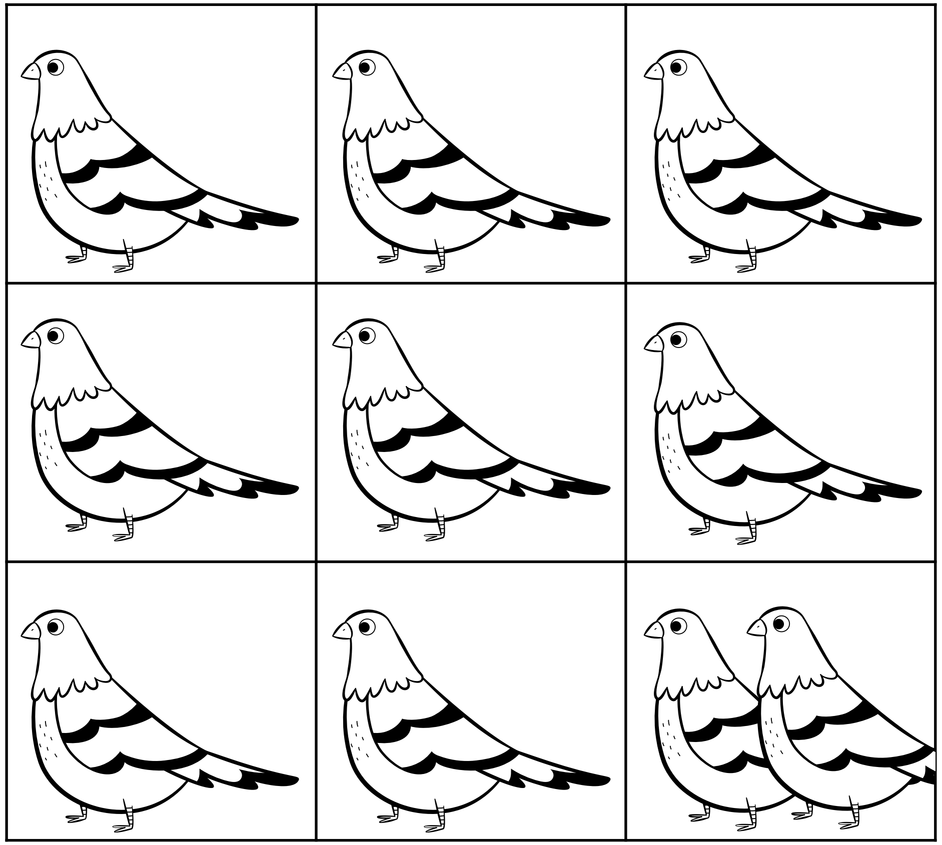
Rephrased:

If n is a positive integer and $n + 1$ or more objects are placed into n boxes, then there is *at least* one box with more than two of the objects.

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Pigeonhole Principle

If you have 10 pigeons and 9 boxes, at least one box will have more than one pigeon.



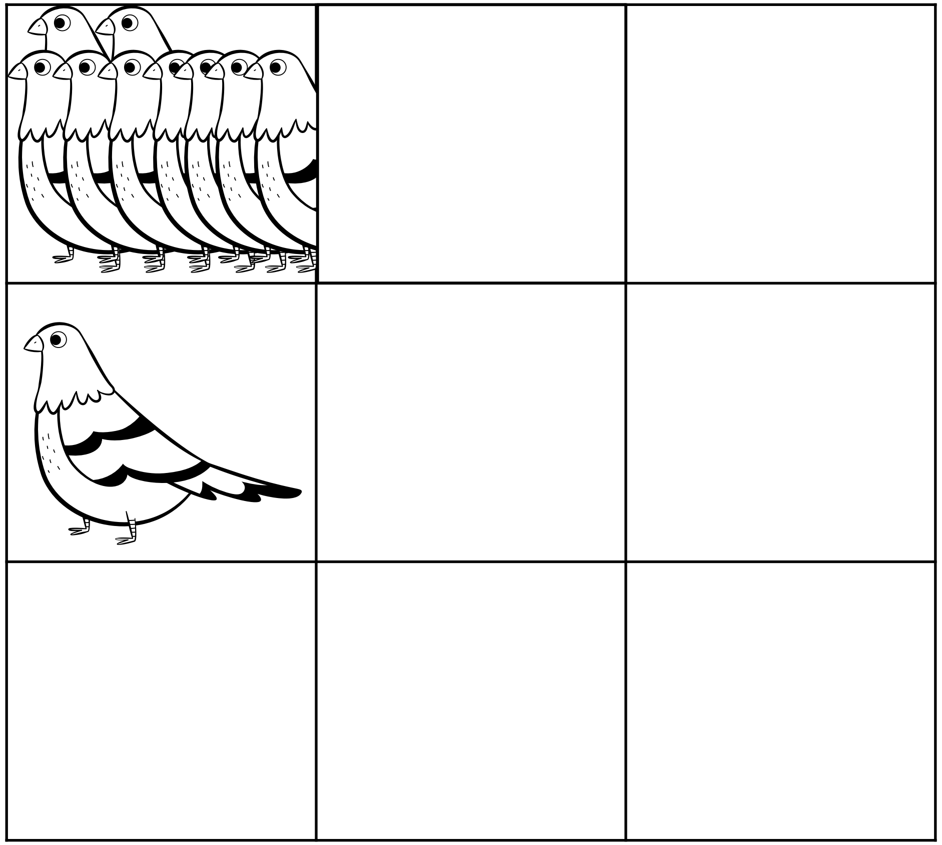
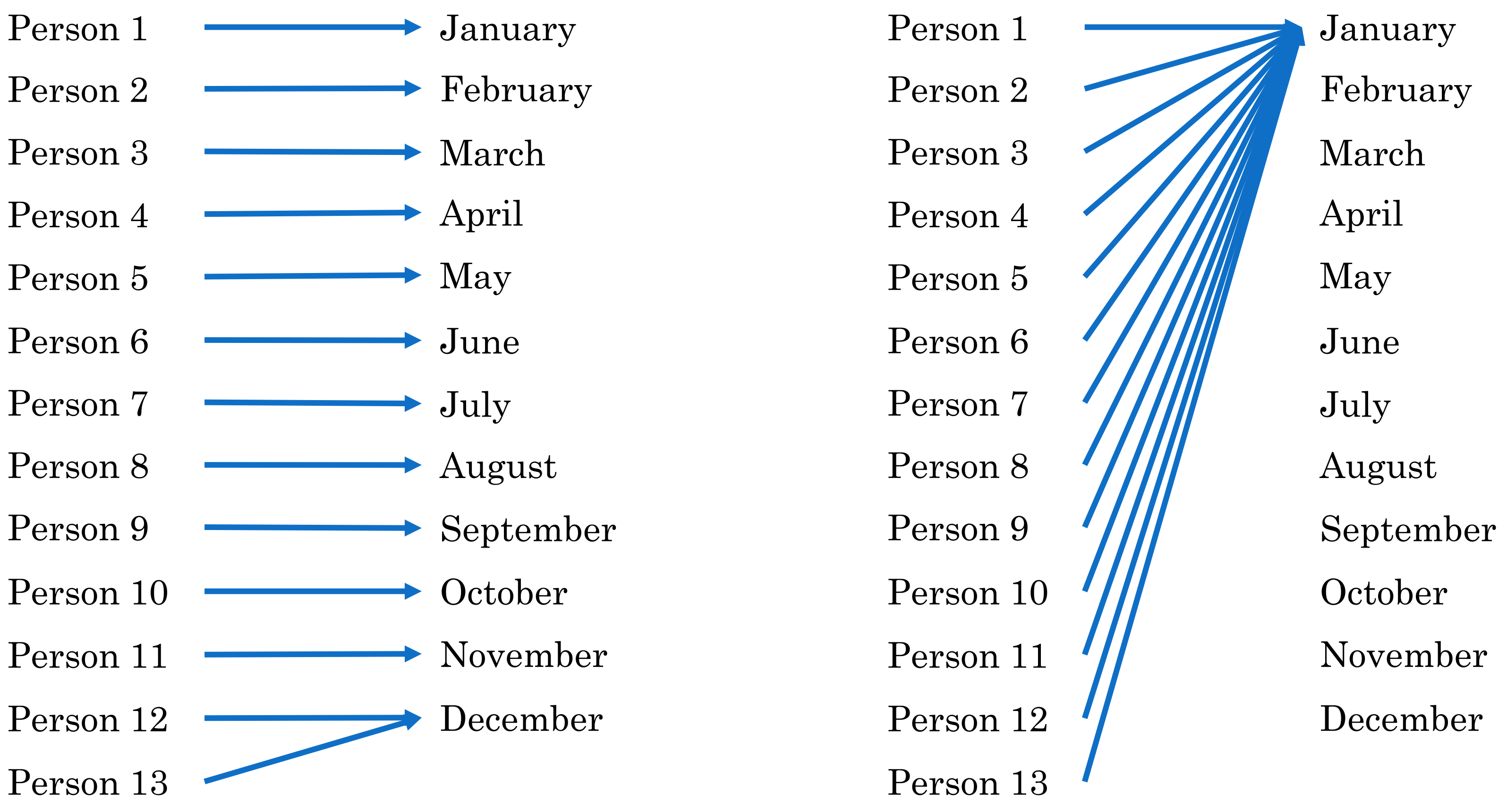


Image by Freepik

Pigeonhole Principle

Among any set of 13 people, there are ***at least*** two people who have a birthday in the same month.



Pigeonhole Principle

Among any set of 367 people, how many of them must have the same birthday?

At least two people must have the same birthday because there are only 366 possible birthdays.

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Pigeonhole Principle

How many students must be in a class to guarantee that at least two students receive the same score on a quiz, if the quiz has possible scores from 0 to 30?

How many possible scores are there? 31

How many students must be in the class to guarantee that two receive the same score?

32

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Generalized Pigeonhole Principle

If a function has n elements in the domain and k elements in the co-domain, then there is an element y in the co-domain such that the function maps at least $\lceil n/k \rceil$ elements in the domain to y.

Example:

Among a group of 25 people, there are at least 3 whose birthday falls in the same month.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25											

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{25}{12} \right\rceil = 3$$

Generalized Pigeonhole Principle

What if we want to guarantee a minimum number of items?

$$n = k(b - 1) + 1$$

k – Number of items in codomain
b – minimum number we want from the domain

Example:

How many gumballs must we choose from a gumball machine with 5 colors of gumballs to guarantee we get at least 4 gumballs of the same color?

red	blue	green	purple	pink
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16				

$$n = k(b - 1) + 1$$

$k = 5$
$b = 4$
$n = 5(4 - 1) + 1 = 16$

Generalized Pigeonhole Principle

Example:

How many socks must we choose from a drawer with 3 colors of socks to guarantee we get at least 2 socks of the same color?

black	blue	grey
1	2	3
4		

$$n = k(b - 1) + 1$$

$k = 3$
$b = 2$
$n = 3(2 - 1) + 1 = 4$

Practice – Area Codes

How many area codes are needed in a state with 25 million people?

Assume a phone number has the form $(nxx)\ nxx - xxxx$ where n is a digit from 2-9 and x is a digit from 0-9.

How many 7-digit phone numbers are possible? $8 * 10^6 = 8{,}000{,}000$

How many area codes do we need? $\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{25000000}{8000000} \right\rceil = 4$