Counting

Counting by Complement

How many 8-bit strings do not begin with 01?

1. How many 8-bit strings are there with no restrictions?

2^8

2. How many 8-bit strings DO begin with 01?

01_ _ _ _ _ _ 2^6

3. Subtract

$2^8 - 2^6 = 192$

Permutations with repetitions

How many ways can you order the letters in the word MATHEMATICS?

Note there are 2 M's, 2 A's, and 2 T's. There are 11 letters total.

The number of distinct sequences with n_1 1's, n_2 2's, ..., n_k k's, where $n = n_1 + n_2 + \cdots + n_k$ is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Therefore, we have $\frac{11!}{2!2!2!1!1!1!1!1!} = \frac{11!}{8} = 4989600$

Permutations with repetitions

$$\frac{n!}{n_{1!}n_{2!}\cdots n_k!}$$

How many bit strings with three 0's and five 1's? 00011111

$$\frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}\,3!} = \frac{8\cdot 7\cdot \cancel{6}}{\cancel{3}\cdot \cancel{2}\cdot 1} = 8\cdot 7 = 56$$

Multisets / Multichoosing

Subsets **with repetition allowed**. Order does not matter.

$S = \{a, b, c, d, e\}$

Some of the ways to choose a subset of three elements:

$\{a, b, c\}$

$\{a, a, a\}$

$\{a, b, b\}$

$\{c, e, e\}$

n = number of items to select
m = number of varieties

$$\binom{n+m-1}{m-1} = \frac{(n+m-1)!}{(m-1)!\,\big(n+m-1-(m-1)\big)!} = \frac{(n+m-1)!}{(m-1)!\,n!}$$

Note:

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

Snack time!

$$\binom{n+m-1}{m-1}$$

How many ways can we select 4 candy bars from a bowl containing Snickers, Twix, and Butterfingers? Assume order does not matter and repetition is allowed. There are at least 4 of each type of candy bar in the bowl.

$n =$ number of items to select
$m =$ number of varieties

$n = 4$ (selecting four candy bars)
$m = 3$ (there are three kinds to choose from)

$$\binom{n+m-1}{m-1} = \binom{6}{2}$$

$$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6*5}{2} = 15$$

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Stars and Bars / 0s and 1s

Whiteboard

Number of Solutions to a Fixed Sum

Count the number of solutions to the equation

$x_1 + x_2 + x_3 = 4$ where x_1, x_2, x_3 are integers.

This is the same problem as selecting 4 items from 3 varieties, where we have x_1 of the first variety, x_2 of the second variety, and x_3 of the third variety. This is the same problem as the candy bar problem.

$n = 4$
$m = 3$

$$\binom{4+3-1}{3-1} = \binom{6}{2}$$

$$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6 * 5}{2} = 15$$

Try it

Count the number of solutions to the equation

$x_1 + x_2 + x_3 = 15$

Now we are selecting 15 items from 3 varieties, where we have x_1 of the first variety, x_2 of the second variety, and x_3 of the third variety.

$n = 15$
$m = 3$

$$\binom{15+3-1}{3-1} = \binom{17}{2}$$

$$\frac{17!}{2!\,15!} = \frac{17 * 16}{2} = 136$$

Handshakes

There are 110 people at a meeting. They each shake hands with everyone else. How many handshakes were there? Is it a permutation or subset, and how many options are there?

Each handshake consists of 2 people.

How many subsets of 2 people are there? This is the number of handshakes that takes place.

This is a subset problem.

$$\binom{110}{2} = 5995$$

Additional Exercises:

10.6.3

10.6.4

10.7.1

10.7.3

10.8.1

10.8.2

10.9.1

10.9.2