Counting

Review

Permutations with repetition

How many ways can you rearrange this binary string 000111?

$$\frac{6!}{3!\,3!} = \frac{720}{36} = 20$$

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Multisets

How many ways can you select five pieces of candy from a bowl with 10 different kinds?

$$\binom{14}{9} = \frac{14!}{5!\,9!} = 2002$$

$$\binom{n+m-1}{m-1}$$

Balls in bins

- Six different types of counting problems, with six different ways to count

	No restrictions	At most one ball per bin	Same number of balls in each bin
	(any positive m and n)	(m must be at least n)	(m must evenly divide n)
Indistinguishable balls	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	1
Distinguishable balls	m^n	$P(m,n)$	$\frac{n!}{((n/m)!)^m}$

Inclusion-exclusion Principle (2 sets)

Let A and B be two finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion-exclusion Principle (3 sets)

Let A, B, C be finite sets. Then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

General Inclusion-exclusion Principle

Let $A_1, A_2, \ldots, A_n$ be a set of n finite sets.

$$
\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| = & \sum_{j=1}^{n} |A_j| \\
& - \sum_{1 \le j < k \le n} |A_j \cap A_k| \\
& + \sum_{1 \le j < k < l \le n} |A_j \cap A_k \cap A_l| \\
\ldots \\
& + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|
\end{aligned}
$$

General Inclusion-exclusion Principle

For 4 sets, A, B, C, D:

$$
\begin{aligned}
&|A \cup B \cup C \cup D| \\
&= |A| + |B| + |C| + |D| \\
&- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
&+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
&- |A \cap B \cap C \cap D|
\end{aligned}
$$

If all 4 sets are the same size:

$$|A \cup B \cup C \cup D| = 4 * |A| - \binom{4}{2} |A \cap B| + \binom{4}{3} |A \cap B \cap C| - |A \cap B \cap C \cap D|$$

Cardinality of Union by Complement

If we want to find the union of multiple sets, sometimes we can count by the inclusion-exclusion principle.

But sometimes, it is easier to count by complement.

1. Determine the cardinality of the universal set, i.e., the set without restrictions
2. Determine the cardinality of the complement of the set you are looking for
3. Subtract: $|U| - complement(|A_1 \cup A_2 \cup \cdots \cup A_n)$