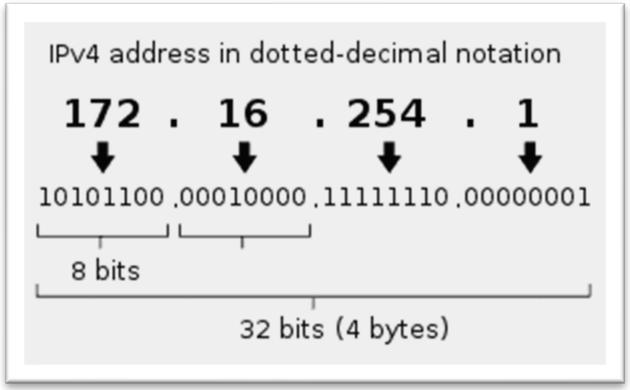
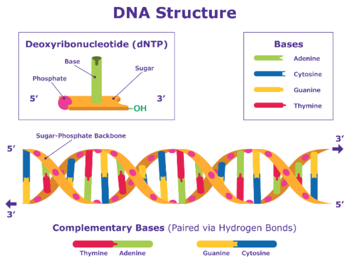
Counting

Learning how to count…again

- Combinatorics – The study of the arrangement of objects
- Enumeration – Counting of objects with certain properties
- First came up in the study of gambling in the 1600s
- Counting objects can be used to solve many problems:
  - How complex is this algorithm?
  - How many telephone numbers needed for a given population?
  - How many IP addresses do we need for a given region?
  - How complex does a password need to be?
  - DNA sequencing





Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Tips

- Play with the problem.
- Try a different approach.
- There is usually more than one way to arrive at the same solution.
- Do as many problems as you can. Try all the additional exercises.

Two basic counting principles

- Product Rule

- Sum Rule

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Product Rule

Product Rule – **both-and**

- Applies when a procedure is made up of multiple different tasks **and each** task must be completed.
- If there are n_1 ways to do the first task and n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

How many bitstrings of length 7? (0000000 to 1111111)

1 or 0

$\underline{}$ $\underline{}$ $\underline{}$ $\underline{}$ $\underline{}$ $\underline{}$ $\underline{}$

2 * 2 2 2 2 2 2 $= 2^7 = 128$

How many passwords of length 8? (Including uppercase, lowercase, digits)

$(26 + 26 + 10)^8 = 218340105584896$

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Product Rule - Example

Let's play musical chairs?

Suppose we have a function assigning 4 students to 8 possible seats.

Is this function One-to-one? Onto?

How many ways to assign the seats?

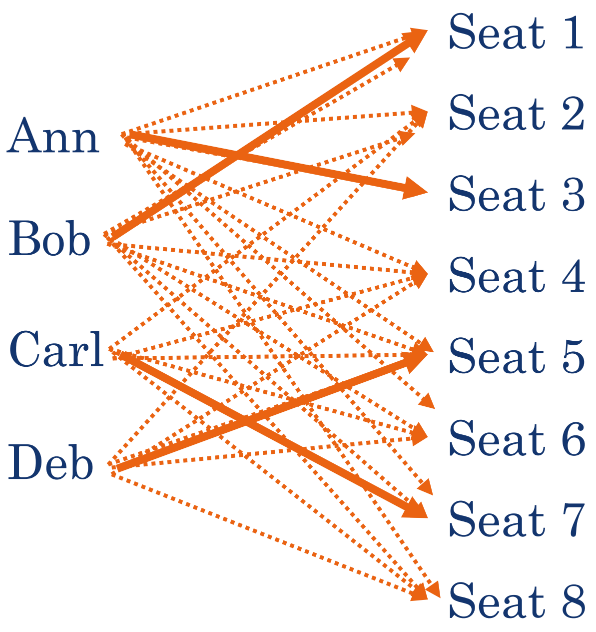
Ann: 8 ways to assign a seat

Bob: 7 remaining ways to assign a seat

Carl: 6 remaining ways to assign a seat

Deb: 5 remaining ways to assign a seat

$$8 * 7 * 6 * 5 = 1680$$



How many?

How many possible bit strings of 8 bits?

$2 * 2 * 2 * 2 * 2 * 2 * 2 * 2 \; = \; 2^8 = \; 256$

How many colors can be represented using rgb, where r, g, and b values are each represented with 8 bits?

$256 * 256 * 256 = 256^3 = 16{,}777{,}216$

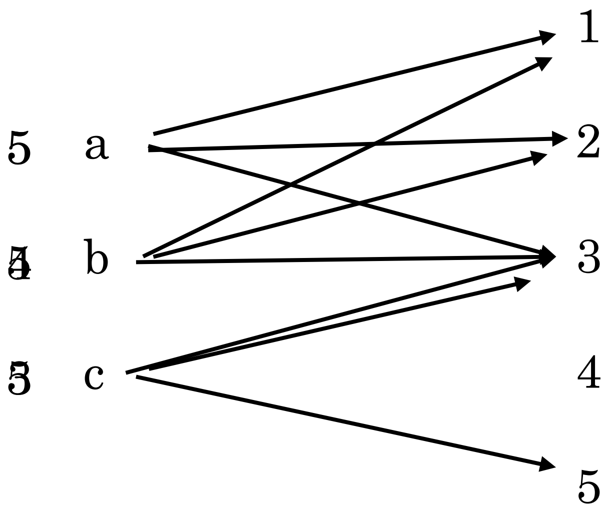
How many colors with rgba?

$256^4 = 4{,}294{,}967{,}296$

4 blocks of 8 bits each is 32 bits. $2^{32} = 4{,}294{,}967{,}296$

How many functions?

How many possible different functions map from a set with three elements to a set with five elements?



$5 * 5 * 5 = 5^3 = 125$

How many different **one-to-one** functions?

$5 * 4 * 3 = 60$

How many different **onto** functions?

None, there are only three elements in the domain but five in the codomain.

range ≠ codomain

Sum Rule

Sum Rule – **either-or**

- If a task can be done in **either** n_1 **or** n_2 ways, where none of the n_1 ways overlaps with the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Examples:

How many ways to choose one letter if there are 5 vowels and 21 consonants?

5 + 21 = 26 (Choose one vowel OR one consonant)

How many ways to choose one vowel AND one consonant?

5 * 21 = 105

Suppose you can choose a final project from one of three lists. The first list contains 10 different projects, the second list contains 12 different projects, and the third list contains 5 different projects. How many possible projects are there to choose from?

10 + 12 + 5 = 27

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Practice

How many possible license plates are there given the following constraints:

Each character can be a capital letter or a digit except for 0 and 1.

Must have 7 characters.

Must begin with a letter.

Examples:

M239ABC
AD8Y7D4
Q893262

___	___	___	___	___	___	___
26	34	34	34	34	34	34

$$26 * 34 * 34 * 34 * 34 * 34 * 34 = 26 * 34^6 = 40{,}164{,}914{,}816$$

r-Permutations vs r-Combinations (Subsets)

n: Number of things to choose from
r: How many things to choose

Order matters

$$P(n,r) = \frac{n!}{(n-r)!}$$

Think sequence or arrangement

Order does not matter

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

"n choose r"

Think sets

r-Permutations

Order matters

Example:

How many ways can we select three students to stand in line for a picture from a group of five students?

5 * 4 * 3 = 60

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r-Permutations

Does order matter?

Let's come up with a formula:

$S = \{a, b, c, d, e\}$

How many ways can we choose r students from S to **stand in line**?

$n = 5, r = ?$

Choose 3:

$P(5,3) = 5 * 4 * 3 = 60$

$$\frac{5 * 4 * 3 * \cancel{2} * \cancel{1}}{\cancel{2} * \cancel{1}} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{120}{2} = 60$$

Choose 2:

$\mathrm{P}(5,2) = 5 * 4 = 20$

$$\frac{5 * 4 * \cancel{3} * \cancel{2} * \cancel{1}}{\cancel{3} * \cancel{2} * \cancel{1}} = \frac{120}{6} = 20$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

r-Combinations (AKA Choosing AKA Subsets)

Order does not matter

Example:

There are 4 students at a table. We need to form a party planning committee of three students from that table. How many different possible committees are there?

$\{ann, bob, carl, deb\}$

$\{bob, carl, deb\}, \{ann, carl, deb\}, \{ann, bob, deb\}, \{ann, bob, carl\}$

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Combinations

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

"n choose r"

$S = \{a, b, c, d, e\}$

Try listing out all the possible subsets with three elements. How many are there?

$\{a, b, c\}$ $\{b, c, d\}$ $\{c, d, e\}$

$\{a, b, d\}$ $\{b, c, e\}$

$\{a, b, e\}$ $\{b, d, e\}$

$\{a, c, d\}$

$\{a, c, e\}$

$\{a, d, e\}$

There are 10

$n = 5$
$r = 3$

$$C(5,3) = \frac{5!}{3!2!} = \frac{120}{12} = 10$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

A group of 30 people have been trained as astronauts for the first mission to Mars. How many ways are there to select a crew of 6 people (assuming all 30 people are interchangeable)?

This is the same thing as the number of subsets of 6 elements from a set of 30 elements. Since order does not matter, we use Combinations.

$$\binom{n}{k}$$

$$C(30,6) = \binom{30}{6} = \frac{30!}{6!\,24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593{,}775$$

Python

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Jury Duty Time!

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Imagine it is determined that the ideal jury consists of 5 men and 7 women. There are 18 men and 16 women in the jury pool.

How many possible juries are there?

$$\binom{m}{r}\binom{w}{s}$$

$$\binom{18}{5}\binom{16}{7}$$

98,017,920

$$\frac{18!}{5!\,13!} = \frac{18*17*16*15*14}{5*4*3*2*1} = 8568$$

$$\frac{16!}{7!\,9!} = \frac{16*15*14*13*12*11*10}{7*6*5*4*3*2*1} = 11440$$

Additional Exercises:

10.3.1

10.3.2

10.3.3

10.3.4

10.4.2

10.4.3

10.5.1

10.5.2

10.5.3

Dealing with large factorials

$$\frac{33!}{30!\,3!} = \frac{33 * 32 * 31 * \cancel{30!}}{\cancel{30!} * 3!} = \frac{\overset{11}{\cancel{33}} * \overset{16}{\cancel{32}} * 31}{\cancel{3} * \cancel{2}} = 11 * 16 * 31 = 5456$$

Much simpler!