Relations

Cartesian Product Review

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Example:

$A = \{1, 2\}$

$B = \{a, b, c\}$

What is $A \times B$? $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

What is $B \times A$? $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

Note: $A \times B \neq B \times A$

Functions Review

A function f from A to B is an assignment of *exactly one* element of B to each element of A:

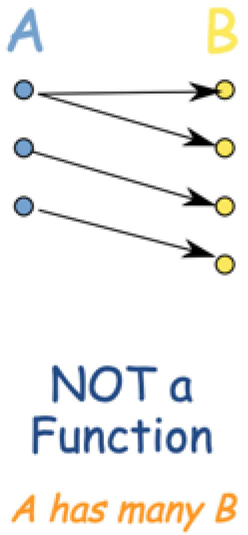
$f(a) = b$ where b is the unique element of B assigned to the element a of A

$A = \{1,2,3\}$

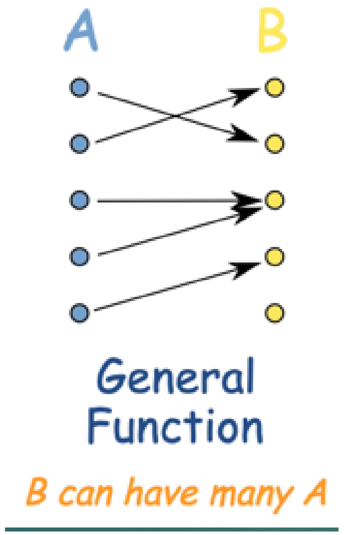
$B = \{2,4,6\}$

Function: $\{(1,2), (2,4), (3,6)\}$ $\quad b = 2a$

Not Function: $\{(1,2), (1,4), (1,6)\}$ Student ID to Class ID



This is called a **Relation**



Function
This is a special kind of **relation**.

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Relation

- Set of ordered pairs (a, b) where $a \in A$ and $b \in B$
- Used to represent some kind of relationship between elements of A and elements of B
- A relation from A to B is a **set** R of **ordered pairs** where the first element is from A and the second element is from B

$$a \, R \, b$$

$$(a, b) \in R$$

Example of **a** relation R from set A to set B where $A = \{a, b, c\}$ and $B = \{1,2,3\}$

$\mathrm{R} = \{(a, 1), (a, 3), (b, 1), (c, 1), (c, 2)\}$

- A **relation** between A and B is a **subset** of the Cartesian product of A and B.

$$a \, R \, b \rightarrow R \subseteq A \times B$$

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Example:

We can represent a set of ordered pairs as a 2-D Matrix, where a 0 or 1 indicates whether a given (row, column) is in the set:

$$\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$$

	1	2	3
a	1	1	1
b	1	1	1
c	1	1	1

Example:

$$\{(a,1),(a,3),(b,1),(c,2),(c,3)\}$$

	1	2	3
a	**1**	**0**	**1**
b	**1**	**0**	**0**
c	**0**	**1**	**1**

Example:

Given the sets $A = \{a, b, c\}$ and $B = \{1,2,3\}$, which of the following is a valid relation R from A to B?

a. {(a,1), (1,a)}

b. {(a,1), (a,2), (b,2)}

c. {(1,a), (2,a), (3,a)}

d. {(b,1), (b,3), (c,1)}

b.

	1	2	3
a	**1**	**1**	**0**
b	**0**	**1**	**0**
c	**0**	**0**	**0**

d.

	1	2	3
a	**0**	**0**	**0**
b	**1**	**0**	**1**
c	**1**	**0**	**0**

Properties of Relations

- Reflexive
- Anti-reflexive
- Symmetric
- Anti-symmetric
- Transitive

reflexive:	$\forall x \in A, xRx$
anti-reflexive:	$\forall x \in A, \neg(xRx)$
symmetric:	$\forall a \forall b, aRb \rightarrow bRa$
anti-symmetric:	$\forall a \forall b, aRb \land bRa \rightarrow a = b$
transitive:	$\forall a \forall b \forall c, aRb \land bRc \rightarrow aRc$

Reflexive

A relation is reflexive if **every** element of A is related to itself.

$$\forall a \in A\left((a, a) \in R\right)$$

$$\forall a \in A\ aRa$$

Example:

R = set of all people (x, y) with the same mother and father.

xRx for every person x

" You have the same mother and father as yourself "

What are some mathematical relations that are reflexive?

Reflexive math relations:

$=$	$\forall x\ x = x$
$\geq$	$\forall x\ x \geq x$
$\leq$	$\forall x\ x \leq x$

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Given the set $A = \{a, b, c\}$, which of the following relations on A are reflexive?

a. {(a,a), (a,b), (a,c)} Not reflexive. Missing (b,b) and (c,c)

b. {(a,a), (a,b), (b,b), (c,c)}

c. {(a,a), (a,c), (b,b), (b,c), (c,a),(c,c)}

d. {(a,b),(b,c), (c,a)} Not reflexive. Missing (a,a), (b,b), and (c,c)

Anti-reflexive

A relation is anti-reflexive if **every** element of A is **not** related to itself.

$$\forall a \in A \left((a, a) \notin R\right)$$

$$\forall a \in A \neg(aRa)$$

Every element in the set is not related to itself.

Example:

R = set of all people (x, y) where x is younger than y.

$\neg xRx$ for every person x

"You are not younger than yourself."

Anti-reflexive math relations:

$<$ $\quad \forall x \neg(x < x)$
$>$ $\quad \forall x \neg(x > x)$

Anti-reflexive

A relation is anti-reflexive if **every** element of A is **not** related to itself.

$$\forall a \in A\ \big((a, a) \notin R\big)$$

$$\forall a \in A\ \neg(aRa)$$

Every element in the set is not related to itself.

Example:

R = set of all people (x, y) where x is grandparent to y.

$\neg xRx$ for every person x

"You are not your own grandparent." (except for this guy)

Given the set $A = \{a, b, c\}$, which of the following relations on A are anti-reflexive?

a. {(a,a), (a,b), (a,c)}

Not anti-reflexive. Contains (a,a)

b. {(a,b), (b,c), (a,c)}

c. {(a,c), (c,a), (b,c), (b,a),(c,b)}

d. {(a,b),(b,c),(c,a),(c,c)}

Not reflexive. Contains (c,c)

Symmetric

A relation is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

$$\forall a \forall b \big((a, b) \in R \rightarrow (b, a) \in R\big)$$

$$\forall a \forall b \ (aRb \rightarrow bRa)$$

If a is related to b, then b is related to a.

Example:
R = set of all people (x, y) where x and y are siblings.

If Bob is Sue's sibling, then Sue is Bob's sibling.

Symmetric math relations:
$=$
if $x = y$ then $y = x$

Which of these relations are symmetric?

a. $R = \{(1,1), (1,2), (2,1)\}$ Yes, symmetric

b. $R = \{(1,1), (2,2), (2,3), (3,2)\}$ Yes, symmetric

c. $R = \{(1,2)\}$ No, not symmetric. Missing $(2,1)$

d. $R = \{(1,1), (1,2), (1,3), (2,1)\}$ No, not symmetric. Missing $(3,1)$

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Anti-symmetric

A relation is anti-symmetric if for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

$$\forall a \forall b \left(\left((a, b) \in R \land (b, a) \in R \right) \to (a = b) \right)$$

$$\forall a \forall b \left((aRb \land bRa) \to (a = b) \right)$$

Another way of thinking about it:

Anti-symmetric means the relation does not contain **any** symmetric pairs

Which of these relations are anti-symmetric?

a. $R = \{(1,1), (1,2), (2,1)\}$ — No, contains (2,1) and (1,2)

b. $R = \{(1,1), (2,2), (2,3), (3,2)\}$ — No, contains (2,3) and (3,2)

c. $R = \{(1,2)\}$ — Yes. Anti-symmetric

d. $R = \{(1,1), (1,2), (1,3), (2,2), (2,3)\}$ — Yes. Anti-symmetric

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Transitive

A relation is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

$$\forall a \forall b \forall c \left(\left((a, b) \in R \land (b, c) \in R\right) \rightarrow (a, c) \in R\right)$$

$$\forall a \forall b \forall c \left((aRb \land bRc) \rightarrow aRc\right)$$

If a is related to b and b is related to c, then a is related to c.

Example:
Same parent as.
Bob has same parent as Sue and Sue has the same parent as Bill. Bob has the same parent as Bill.

Which of these relations are transitive?

a. $R = \{(1,1), (1,2), (2,1)\}$ No, missing (2,2)

b. $R = \{(1,1), (2,2), (2,3), (3,2)\}$ No, missing (3,3)

c. $R = \{(1,2)\}$ Yes. Transitive

d. $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$ Yes. Transitive

Transitive math relations:

$<$ $(1 < 2 < 3) \rightarrow (1 < 3)$

$>$

$\leq$

$\geq$

$=$

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Profile a binary relation using the five properties:

reflexive:	$\forall x \in A\ xRx$
anti-reflexive:	$\forall x \in A, \neg(xRx)$
symmetric:	$\forall a \forall b\ aRb \rightarrow bRa$
anti-symmetric:	$\forall a \forall b\ aRb \wedge bRa \rightarrow a = b$
transitive:	$\forall a \forall b \forall c\ aRb \wedge bRc \rightarrow aRc$

$A = \{1,2,3,4,5\}$

$R = \{(a, b) \mid a + b = 6\}$ (set builder notation)

$R = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$ (roster notation)

Which of the following properties apply to R?

a. Reflexive

b. Anti-reflexive

c. Symmetric

d. Anti-symmetric

e. Transitive

Directed Graphs

Directed Graph definition:

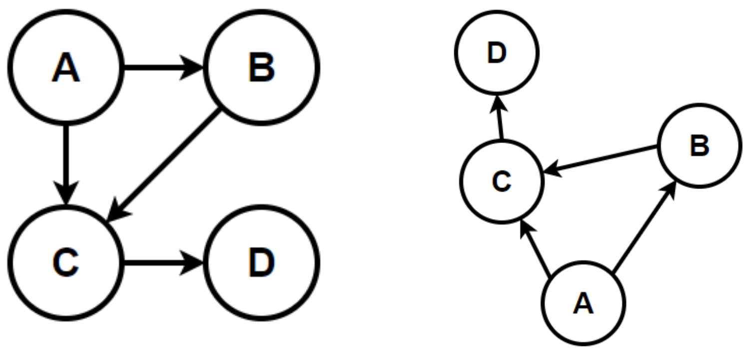
$G = (V, E)$ where V is a nonempty set of vertices and E (the *edges)* is a set of ordered pairs of elements of V

Example:

$G = (\{A, B, C, D\}, \{(A, B), (B, C), (A, C), (C, D)\})$

Note that E is a subset of $V \times V$. In other words, a directed graph is the same as a binary relation on the set V

It can be helpful to draw it, but this is not necessary to be considered a graph

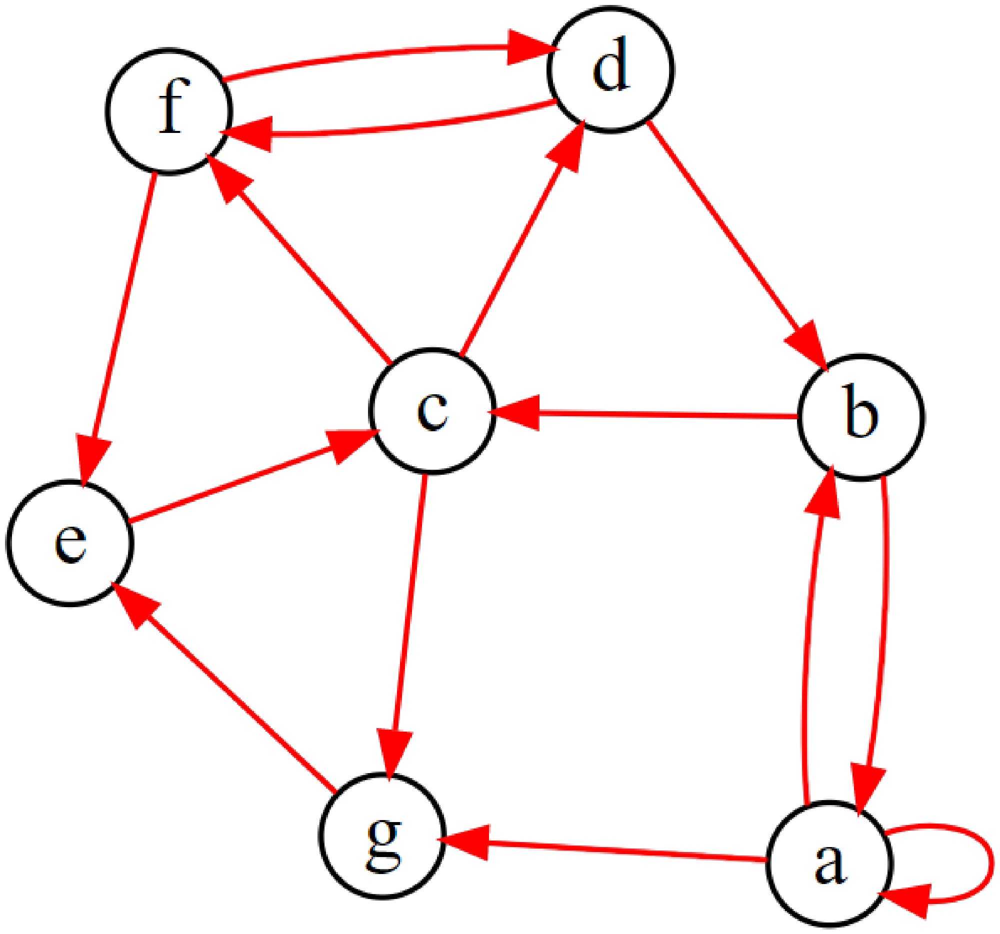


These both represent the same graph

Directed Graphs

With a partner, discuss the following terms using this graph.

What is the **in-degree** of vertex **a**?
What is the **out-degree** of vertex **c**?
What is the **head** of edge (b,c)?
What is the **tail** of edge (e,c)?
Which vertex has a **self-loop**?
Find a **trail**. (no edge occurs more than once)
Find a **path**. (no vertex occurs more than once)
Find a **circuit**. (no edge occurs more than once)
Find a **cycle**. (no edge or vertex occurs more than once)



How do directed graphs show the properties of binary relations?

- Reflexive
- Anti-reflexive
- Symmetric
- Anti-symmetric
- Transitive
- Equivalence relation

reflexive:	$\forall x \in A, xRx$
anti-reflexive:	$\forall x \in A, \neg(xRx)$
symmetric:	$\forall a \forall b\ aRb \ \to bRa$
anti-symmetric:	$\forall a \forall b\ aRb \wedge bRa \ \to a = b$
transitive:	$\forall a \forall b \forall c\ aRb \wedge bRc \to aRc$

Composition of Relations

Let R be a relation from set A to B.

Let S be a relation from set B to C.

$S \circ R$ is the set of pairs (a, c) where $a \in A$ and c $\in C$ and where $(a, b) \in R$ and $(b, c) \in S$.

Example:
A: $\{b, c, d\}$ B: $\{b, c, d, e\}$ C: $\{a, b, c\}$

R is a relation on A to B: $\{bb, be, cd, db, de\}$
S is a relation on B to C: $\{ba, ca, db, dc, eb\}$

To find $S \circ R$, find elements that are the second element of a pair in R and are the first element of a pair in S.

What is $S \circ R$?

$\{ba, bb, cb, cc, da, db\}$

Composition of a Relation with Itself

Sometimes we want to compose a relation with itself. When might this be useful?

Let R be a relation on the set of all people such that $(x, y) \in R$ if x is a parent of y.

Then $(x, z) \in R \circ R$ iff there is an x and y such that $(x, y) \in R$ and $(y, z) \in R$.

Thus, person x is the parent of person y and y is the parent of z.

What is the relationship between x and z?

x is the grandparent of z

$R \circ R$ is also written as R^2

R^2 represents all walks of length 2 in a directed graph, or all second-order relations.

Powers of a Relation

R^n is defined recursively as:

$R^1 = R$

$R^{n+1} = R^n \circ R$

R^n represents all walks of length n

$R^1 = R$
$R^2 = R \circ R$
$R^3 = R^2 \circ R = (R \circ R) \circ R$
$R^4 = R^3 \circ R = \big((R \circ R) \circ R\big) \circ R$
$\ldots$

Represents all walks of length 2
Represents all walks of length 3
Represents all walks of length 4

Example

$R = \{(1,1), (2,1), (3,2), (4,3)\}$

Find R^2, R^3, R^4

$R = \{(1,1), (2,1), (3,2), (4,3)\}$
$R = \{(1,1), (2,1), (3,2), (4,3)\}$

$R^2 = R \circ R$

$R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$

$R^3 = R^2 \circ R$

$R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = R^3 \circ R$

$R^4 = \{(1,1), (2,1), (3,1), (4,1)\}$

Note that $R^n = R^3$ for any $n = 4, 5, 6, \ldots$

Matrix vs properties

How can we use a matrix to determine the properties of a relation?

Reflexive

Anti-reflexive

Symmetric

Anti-symmetric

Equivalence Relations
&
Relational Databases

Quick Review

How do directed graphs show the properties of binary relations?

How do matrices show the properties of binary relations?

- Reflexive
- Anti-reflexive
- Symmetric
- Anti-symmetric
- Transitive

reflexive:	$\forall x \in A, xRx$
anti-reflexive:	$\forall x \in A, \neg(xRx)$
symmetric:	$\forall a \forall b \; aRb \; \rightarrow bRa$
anti-symmetric:	$\forall a \forall b \; aRb \wedge bRa \; \rightarrow a = b$
transitive:	$\forall a \forall b \forall c \; aRb \wedge bRc \rightarrow aRc$

Equivalence Relation

- A binary relation that is
  - Reflexive
  - Symmetric
  - Transitive

$a \sim b$ means "a is equivalent to b"

- the **same** ______ as
  - The same parent as
  - The same name as
  - The same birthday as

All equivalent variable names in C:

```
number_of_named_tropical_storms
number_of_named_tropical_storms_atlantic
number_of_named_tropical_storms_pacific
number_of_named_tropical_storms_2020
number_of_named_tropical_storms_2021
```

C variables – the same first 31 characters

Key Point: Whenever you have a relation where something has **the same ______** as something else, you have an equivalence relation (usually)

- x has the same age as y
- the same birthday as
- the same name as
- the same major as
- the same first three bits as
- the same first 31 characters as
- the same ... as

Equivalence Classes

If A is the domain of an equivalence relation R and $a \in A$ then $[a]_{\mathrm{R}}$ is the set of all $x \in A$ such that $a{\sim}x$.

$$[a]_{\mathrm{R}} = \{x \mid (a, x) \in \mathrm{R}\}$$

a *is equivalent to* x

$$[a]_{\mathrm{R}} = \{x \mid a{\sim}x\}$$

If there is an *equivalence relation* R on set A, the set of **all** elements x related to an element $a \in A$ is the *equivalence class*.

Example: R = set of all people (x, y) where x has the same last name as y

Reflexive: $x\mathrm{R}x$ for every person x. All people have the same last name as themself.
Symmetric: $y\mathrm{R}x$ whenever $x\mathrm{R}y$ for all (x, y). If you have the same last name as Bob, then Bob has the same last name as you.
Transitive: $(xRy \wedge yRz) \rightarrow xRz$. If person 1 has the same last name as person 2 and person 2 has the same last name as person 3, then person 1 has the same last name as person 3.

[Smith] = The set of all people with the last name of "Smith".
[x] = The set of all people with the last name of x.

Example:

Let A be the set of all students in the class.

Let R be the equivalence relation on A represented by all pairs (x, y) where x and y are at the same table.

List out the pairs (x, y) for all students in the class.

{(bob, sam), (sam, bob), (bob, sue), (sue, bob), (sue, sam), (sam, sue), (bob, bob), (sam, sam), (sue, sue), (bill, jeff), (jeff, bill), (jeff, jeff), (bill, bill), ...}

What is [bob]?

[bob] = {bob, sam, sue}

This subset of A consists of all students at the same table as **bob**. This is the set of all students equivalent to **bob** with respect to $\boldsymbol{R}$. This subset is an **equivalence class** of the relation.

Example

Domain: All people

$R = \{(x, y) |\ x$ and y have the same mother$\}$

Is this an equivalence relation?

Reflexive? Yes. You have the same mother as yourself.

Symmetric? Yes. If Bob has the same mother as Sue, then Sue has the same mother as Bob.

Transitive? Yes. If Bob has the same mother as Sue and Sue has the same mother as Bill, then Bob has the same mother as Bill.

What is an equivalence class in this relation?

Each equivalence class is the set of people with the same mother. The equivalence classes form a partition on the set of all people.

Example

Domain: All people

$R = \{(x, y) |\ x$ and y have the same parent$\}$

Is this an equivalence relation?

Reflexive?	Yes. You have the same parent as yourself.
Symmetric?	Yes. If Bob has the same parent as Sue, then Sue has the same parent as Bob.
Transitive?	No. Bob and Sue can have the same mother. Sue and Bill can have the same father. That does not imply that Bob and Bill have the same father.

It is not an equivalence relation.

Example

Domain: $\mathbb{Z}$

$R = \{(x, y) \mid (x, y) \text{ have the same remainder when divided by 5}\}$

Is this an equivalence relation?

Reflexive?	Yes. x has the same remainder as itself.
Symmetric?	Yes. If x as the same remainder as y, then y has the same remainder as x.
Transitive?	Yes. If x has the same remainder as y and y has the same remainder as z, then x has the same remainder as z.

What are the equivalence classes for this relation?

$[0]_R$: $\{-10, -5, 0, 5, 10, 15, 20, \dots\}$

$[1]_R$: $\{-9, -4, 1, 6, 11, 16, 21, \dots\}$

$[2]_R$: $\{-8, -3, 2, 7, 12, 17, 22, \dots\}$

$[3]_R$: $\{-7, -2, 3, 8, 13, 18, 23, \dots\}$

$[4]_R$: $\{-6, -1, 4, 9, 14, 19, 24, \dots\}$

Note that the equivalence classes for this relation form a **partition** of $\mathbb{Z}$.

Relational Databases

name	email	phone	job title	manager	hometown
Bob	bob@company.com	867-5309	Computer Scientist	Sue	Rexburg
Sue	sue@company.com	555-5555	Senior Computer Scientist	John	Idaho Falls
Carl	carl@company.com	555-5432	Engineer	Sue	Rexburg

- Selection
- Projection

In SQL (Structured Query Language):

```
SELECT name, phone, email FROM employees WHERE hometown="Rexburg";
```

Which part is the **Selection**, and which is the **Projection**?

Bob	867-5309	bob@company.com
Carl	555-5432	carl@company.com

In Python:

```
[(x[0], x[2], x[4]) for x in employees if x[5]=="Rexburg"]
```

Relational Databases

Activity

https://tinyurl.com/cse280-relations