Functions

Quick Review: Cartesian Product

Given these sets:

$A = \{a\}$
$B = \{b, c\}$
$C = \{a, b, d\}$

Write the elements of the following:

1. $A \times (B \cup C)$

$(B \cup C) = \{a, b, c, d\}$

$A \times (B \cup C) = \{a\} \times \{a, b, c, d\} = \{aa, ab, ac, ad\}$

2. $\mathrm{P}(A \times B)$

$A \times B = \{ab, ac\}$

$\mathrm{P}(A \times B) = \{\emptyset, \{ab\}, \{ac\}, \{ab, ac\}\}$

Quick Review: Bit strings

$\{0,1\}^n$ represents the set of all bitstrings of length n.

For example:

$\{0,1\}^2 = \{00, 01, 10, 11\}$

$\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

What is the cardinality of $\{0,1\}^6$?

Functions

Also called "mappings" or "transformations"

domain – the set of all possible inputs

codomain or target – the set of all possible outputs

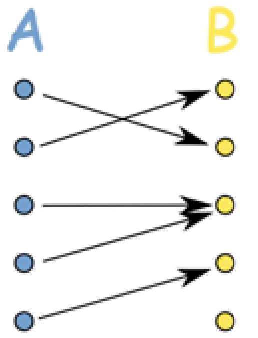
If f is a function from set A to set B, we write:

$$f: A \to B$$

We can also express this relationship in terms of elements of each set:

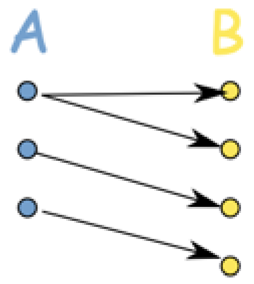
A function f from A to B is an assignment of *exactly one* element of B to each element of A:

$f(a) = b$ where b is the unique element of B assigned to the element a of A



General Function

B can have many A



NOT a Function

A has many B

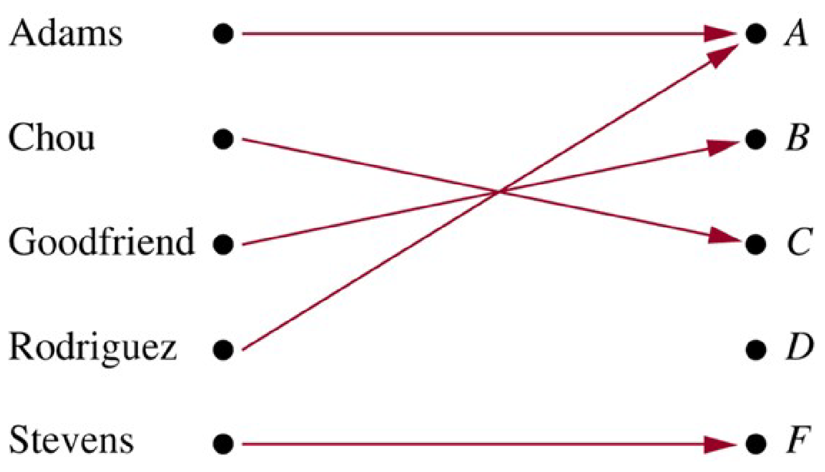
Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Functions

domain – the set of all possible inputs

codomain – the set of all possible outputs

range or image – Those members of the codomain that are actually used.



Example:

Function assigning grades in a discrete math class.

- What is the domain?
  {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- What is the codomain?
  {A, B, C, D, F}
- What is the range?
  {A, B, C, F}

Note: A function $f: A \rightarrow B$ is a subset of $A \times B$

Rosen, Kenneth. *Discrete Mathematics and Its Applications, 7th Edition*. McGraw-Hill, 2012

Functions

Example: $f(x) = x^2$ This function definition is not complete until we define the domain and target

What is the domain and target?

$f: \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$

$f: \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x^2$

```
def square(n: float) -> float:
    return n * n
```

```
def square(n: int) -> int:
    return n * n
```

In Python:

```
def square(n):
    return n * n
```

```
square(2)  -> 4
square(0)  -> 0
square(-1) -> 1
square(π)  -> 9.869604...
```

What is the domain?

All Real numbers ($\mathbb{R}$)? `int`, `float`, ?

What is the codomain?

All positive Real numbers along with 0 ($\mathbb{R}^+ \cup \{0\}$)

`int`, `float`, ?

Example

Floor and Ceiling

floor: $\mathbb{R} \rightarrow \mathbb{Z}$, where $\mathbf{floor}(\boldsymbol{x})$ is the largest integer y such that $y \leq x$

$\text{floor}(x) = \lfloor x \rfloor$ Example: $\text{floor}(1.8) = \lfloor 1.8 \rfloor = 1$

ceiling: $\mathbb{R} \rightarrow \mathbb{Z}$, where $\mathbf{ceiling}(\boldsymbol{x})$ is the smallest integer y such that $y \geq x$

$\text{ceiling}(x) = \lceil x \rceil$ Example: $\text{ceiling}(1.8) = \lceil 1.8 \rceil = 2$

Calculate each of the following:

a. $\lfloor 2.9 \rfloor$ 2

b. $\lceil 17.005 \rceil$ 18

c. $\lfloor -3.5 \rfloor$ -4

d. $\lceil -0.05 \rceil$ 0

In Python:

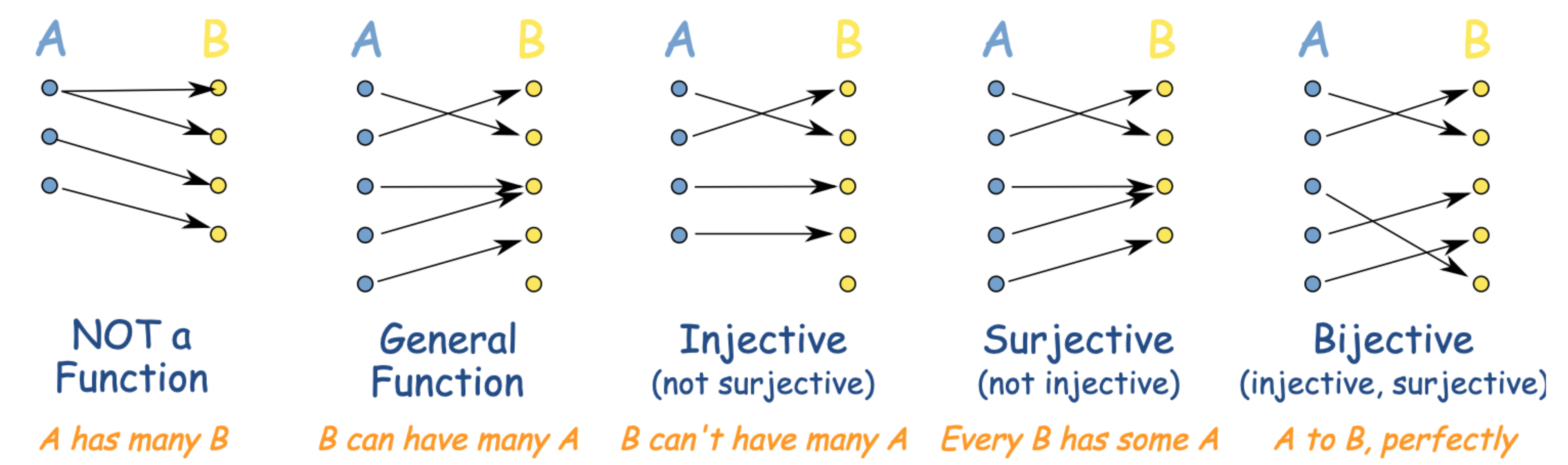
```
from math import floor, ceil

floor(-3.5) # -4

ceil(-0.5)  # 0
```

Example

Describing Functions



one-to-one

onto

both one-to-one and onto

Classify these functions

Are the following functions one-to-one (injective), onto (surjective), neither, or both (bijective)?

a) $f: \mathbb{Z} \to \mathbb{Z}. f(x) = x + 1$

   One-to-one and onto, which is a bijection

b) $f: \mathbb{Z} \to \mathbb{Z}. f(x) = 3x$

   One-to-one but not onto

c) $f: \mathbb{R} \to \mathbb{R}. f(x) = 3x$

   One-to-one and onto, which is a bijection

d) $f: \{0,1\}^3 \to \{0,1\}^3$. The output is obtained by reversing the input. For example, $f(011) = 110$

   One-to-one and onto, which is a bijection

e) $f: \{0,1\}^3 \to \{0,1\}^2$. The output is obtained by dropping the first bit. For example, $f(011) = 11$

   Onto but not one-to-one

Inverse of a Function

A function f has an inverse **if and only if f is a bijection**.

The inverse can be obtained by swapping the items in each pair in f. $f^{-1} = \{(y, x): (x, y) \in f\}$

Example:
$f: \mathbb{Z} \to \mathbb{Z}. f(x) = x + 9$

Is f a bijection?

Yes.

What is $f^{-1}(x)$?

$f^{-1}(x) = x - 9$

Example:
$f: \mathbb{Z} \to \mathbb{Z}. f(x) = 2x + 1$

Is f a bijection?

$f(0) = 1$
$f(1) = 3$
$f(2) = 5$
$f(3) = 7$
…

No, f is not a bijection, so inverse of f is not well-defined.

Exponents and Logarithms

You want to share a bag of Skittles with your friends. The bag has 33 Skittles. How many times can you divide your bag of skittles in half if you keep the larger pile each time?

$\lceil \log_2 33 \rceil = 6$

If you keep the smaller pile each time?

$\lfloor \log_2 33 \rfloor = 5$