Logic, Predicates, & Quantifiers

$\forall x$

$\exists! x \, (x = 3)$

$\exists! x \, P(x) \rightarrow \exists x \, P(x)$

$\forall x \, (F(x) \rightarrow C(x))$

$\exists x \, (S(x) \land N(x))$

$\exists x$

De Morgan's Laws

De Morgan’s laws for logic state:

The negation of a conjunction is the disjunction of negations; likewise, the negation of a disjunction is the conjunction of negations.

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Example

Simplify this expression using De Morgan's Laws:

$\neg(\neg p \vee q)$

1. $(\neg\neg p \wedge \neg q)$
2. $(p \wedge \neg q)$

$\neg(\neg p \vee q) \equiv (p \wedge \neg q)$

You try it

Negate the following compound proposition by adding parenthesis around the entire expression, negating it, then apply De Morgan's Laws

$(\neg p \wedge \neg q) \vee (p \wedge \neg r)$

Find a match

A. $(p \wedge q) \vee (\neg p \wedge r)$

B. $(p \vee q) \wedge (\neg p \vee r)$

C. $(\neg p \vee \neg q) \wedge (\neg p \vee \neg r)$

D. $\neg(p \vee q) \wedge \neg(p \vee \neg r)$

Laws of Propositional Logic

Idempotent laws:	$p \lor p \equiv p$	$p \land p \equiv p$
Associative laws:	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	$(p \land q) \land r \equiv p \land (q \land r)$
Commutative laws:	$p \lor q \equiv q \lor p$	$p \land q \equiv q \land p$
Distributive laws:	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
Identity laws:	$p \lor F \equiv p$	$p \land T \equiv p$
Domination laws:	$p \land F \equiv F$	$p \lor T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \land \neg p \equiv F$ $\neg T \equiv F$	$p \lor \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \lor q) \equiv \neg p \land \neg q$	$\neg(p \land q) \equiv \neg p \lor \neg q$
Absorption laws:	$p \lor (p \land q) \equiv p$	$p \land (p \lor q) \equiv p$
Conditional identities:	$p \to q \equiv \neg p \lor q$	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

Irani, Sandy. *Discrete Mathematics*. Zyante Inc, 2020

Consider the statement:

$$x > 5$$

Is this a proposition?

$$x + 1 > x$$

Is this a proposition?

$$x \text{ is even}$$

Proposition?

Each of these statements has two parts: a **variable** and a **predicate**.

A **predicate** is a T/F statement that depends on one or more **variables**.

$$P(x) = \text{"}x \text{ is even"}$$

Consider the statement:

$$x > 5$$

Let $P(x)$ represent the predicate "x is greater than 5"

Now, we can turn it into a proposition by giving x a value.

What is the truth value of:

$P(4)$?

$P(6)$?

What is a predicate?

- A T/F statement that depends on one or more variables
- A function that returns True or False
- What are the possible inputs (AKA domain)? It depends
- What are the possible outputs (AKA codomain)? {true, false}

True False

- Python Example

```
seq = [0, 1, 2, 3, 5, 8, 13]
result = list(filter(lambda x: x%2 != 0, seq))
```

- What is the predicate?

example

Quantifiers

- Predicates are **not propositions** because they contain **free variables**.

- One way to turn a predicate into a proposition is by assigning values to the variables.

$$P(x)\colon\ x < 5$$
$$P(4) = \text{True}$$

- Another way to create a proposition from a predicate is to assign a **range** of values rather than plugging in specific values.

Quantifiers

- Another way to create a proposition from a predicate is to assign a **range** of values rather than plugging in specific values.

Let $P(x)$ be the predicate "x is greater than 5"

Is $P(x)$ true for ALL values of x? $\forall x\ P(x)$

Is $P(x)$ true for SOME values of x? $\exists x\ P(x)$

Quantifier Symbols

- $\forall$ — Universal quantifier.
  Meaning: For every, Every, All, For all
- $\exists$ — Existential quantifier.
  Meaning: There is at least one, some, for some, at least one, there is
- $\neg\forall$ — Not every
- $\neg\exists$ — There is not one, there are none, there are not any

LaTeX for $\forall$ is **\forall**

LaTeX for $\exists$ is **\exists**

Example:

Let $P(x)$ represent the statement $x + 1 > x$

What is the truth value of:

$\forall x\, P(x)$, where the universe of discourse or **domain** is $\mathbb{R}$

$\mathbb{R}$ = Real Numbers

We can also say:

$\forall x \in \mathbb{R}, P(x)$

Example:

Let $P(x)$ represent the statement $x > 3$

What is the truth value of:

$\forall x\ P(x)$, where the **domain** is $\mathbb{R}$

$\exists x\ P(x)$

Example:

Express this as a quantified statement:

All elephants are gray.

$\forall x\ (\text{Elephant}(x) \rightarrow \text{Gray}(x))$

Some dogs are white.

$\exists x\ (Dog(x) \wedge White(x))$

De Morgan's Laws

Negate the following:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

For quantifiers

$$\neg\forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg\exists x\, P(x) \equiv \forall x \neg P(x)$$

For Nested Quantifiers

$\neg\forall x\, \forall y\, P(x, y) \equiv \exists x\, \exists y\, \neg P(x, y)$	
$\neg\forall x\, \exists y\, P(x, y) \equiv \exists x\, \forall y\, \neg P(x, y)$	
$\neg\exists x\, \forall y\, P(x, y) \equiv \forall x\, \exists y\, \neg P(x, y)$	
$\neg\exists x\, \exists y\, P(x, y) \equiv \forall x\, \forall y\, \neg P(x, y)$	

De Morgan's Laws - Examples

Simplify the following using De Morgan's Laws:

$\neg\exists x\ P(x)$	$\forall x\ \neg P(x)$
$\neg\exists x\ \big(P(x) \wedge Q(x)\big)$	$\forall x\ \big(\neg P(x) \vee \neg Q(x)\big)$
$\neg\forall x\ \big(P(x) \wedge Q(x)\big)$	$\exists x\ \big(\neg P(x) \vee \neg Q(x)\big)$
$\neg\forall x\exists y\ P(x,y)$	$\exists x\forall y\ \neg P(x,y)$
$\neg\exists x\forall y\ \big(P(x,y) \wedge Q(x)\big)$	$\forall x\exists y\ \big(\neg P(x,y) \vee \neg Q(x)\big)$

Let's do a couple of the challenges together.

Challenge 1.9.1

Challenge 1.10.1